Exercise 8.4

Find unbounded function $v(x)$ that satisfies the linear complimentary conditions. Namely,

- $v(x) \geq (K-x)^+$ for all $x \geq 0$
- $rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0$ for all $x \geq 0$
- At least one of the above two inequalities holds with equality for each $x \geq 0$

Proof

Fix L such that

$$0 < L \leq \frac{2rK}{2r+\sigma^2} = L_*.$$

From Exercise 8.3, we know that for constants A, B the following function satisfies the second order inequality with equality.

$$v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx.$$

We find A and B such that

$$\begin{aligned} v(L) &= K - L \\ v'(L) &= -1. \end{aligned}$$

Note that

$$\begin{aligned} v'(x) &= -\frac{2r}{\sigma^2} \cdot Ax^{-\frac{2r}{\sigma^2}-1} + B \\ &= -\frac{2r}{\sigma^2 x} \cdot (v(x) - Bx) + B \end{aligned}$$

Thus,

$$-\frac{2r}{\sigma^2} \cdot (K - L - BL) + BL = -L.$$

Therefore,

$$K - L(1+B) = L(1+B)\frac{\sigma^2}{2r}$$

So

$$1 + B = \frac{2rK}{L(2r+\sigma^2)} = \frac{L_*}{L}$$

By assumption $L \leq L_*$ which yields that $B \geq 0$. Continuing,

$$\begin{aligned} K &= AL^{\frac{-2r}{\sigma^2}} + L(1+B) \\ &= AL^{-\frac{-2r}{\sigma^2}} + \frac{2rK}{2r+\sigma^2} \end{aligned}$$

Thus

$$A = \frac{\sigma^2 K}{2r+\sigma^2} \cdot L^{\frac{2r}{\sigma^2}}$$

Since $A, B \geq 0$, it holds that $v(x) \geq 0$ for all $x \geq 0$. On the other hand, for each $x \geq L$, there exists $x_0 \in (L, x)$ such that

$$v(x) = v(L) + (x - L)v'(x_0)$$

It suffices to show that $v'(x_0) \geq -1$. In that case, $v(x) \geq K - x$ which put together with previous conclusion gives

$$v(x) \geq (K - x)^+ \text{ for all } x \geq L.$$

We note that

$$\begin{aligned} v'(x_0) &= -\frac{2r}{\sigma^2} A x_0^{-\frac{2r}{\sigma^2}-1} + B \\ &\geq -\frac{2r}{\sigma^2} A L^{-\frac{2r}{\sigma^2}-1} + B \\ & v'(L) \\ &= -1. \end{aligned}$$

Note that

$$\begin{aligned} -\frac{2r}{\sigma^2} A x_0^{-\frac{2r}{\sigma^2}-1} \geq -\frac{2r}{\sigma^2} A L^{-\frac{2r}{\sigma^2}-1} &\iff x_0^{\frac{2r}{\sigma^2}+1} \geq L^{\frac{2r}{\sigma^2}+1} \\ &\iff x_0 \geq L. \end{aligned}$$

Denote

$$\tilde{v}(x) = \begin{cases} K - x & \forall x \leq L \\ v(x) & \forall x \geq L \end{cases}$$

By construction, $\tilde{v}$ satisfies the linear complimentary conditions. Here the first bound (resp. second bound) holds with equality for $x \leq L$ (resp. $x \geq L$). Both bounds hold for all $x \geq 0$ based on what we showed above. On the other hand, $v(x)$ is not linear and $L < L_*$, we have that $\tilde{v} \neq v_{L_*}$. Finally,

$$\begin{aligned} v \text{ is bounded} &\iff B = 0 \\ &\iff L = L_* \end{aligned}$$