Exercise 8.5

Price a perpetual American put that pays dividend. Suppose the differential of this asset is

$$\mathrm{d}S(t) = (r-a)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t)$$

Proof

Denote

$$\tau_L := \min\{t \geq 0 : S(t) = L\}$$

The agent exercises once $t = \tau_L$. The value of the put under this strategy is computed as below.

$$v_L(S(0)) = (K-L)\,\tilde{\mathbb{E}}e^{-r\tau_L}$$

The differential of the asset is

$$\mathrm{d}S(t) = (r-a)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t).$$

We have that

$$S(t) = S(0)\exp\left(\sigma\tilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t\right)$$

Letting $x = S(0)$ and assuming $x > L$, it holds

$$\begin{aligned} S(t) = L &\iff \sigma\tilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t = \log\frac{L}{x} \\ &\iff \tilde{W}(t) + \frac{1}{\sigma}\left(r - a - \frac{1}{2}\sigma^2\right)t = \frac{1}{\sigma}\log\frac{L}{x} \\ &\iff -\tilde{W}(t) - \frac{1}{\sigma}\left(r - a - \frac{1}{2}\sigma^2\right)t = \frac{1}{\sigma}\log\frac{x}{L} \end{aligned}$$

Set

$$\mu = r - a - \frac{1}{2}\sigma^2, m = \log\frac{x}{L}.$$

It is emphasized that $m = \log\frac{x}{L} > 0$. Theorem 8.3.2 gives that

$$\tilde{\mathbb{E}}e^{-r\tau_L} = e^{-m\left(\underbrace{\frac{1}{\sigma^2}\mu + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\mu^2 + 2r}}_{:=\gamma}\right)}$$

Thus

$$v_L(S(0)) = (K-L)\,e^{-m\gamma}$$

For $x \leq L$, $v_L(S(0)) = K - x$. We know the following holds if and only if smooth pasting condition is satisfied by $v_{L_*}(x)$.

$$v_{L_*}(x) = \max_L v_L(x)$$

For $x > L$,

$$\begin{aligned} v_L(x) &= (K - L)e^{\gamma \log L - \gamma \log x} \\ &= (K - L) \cdot \frac{L^\gamma}{x^\gamma} \end{aligned}$$

Thus

$$\begin{aligned} v'(L+) &= -(K - L)\gamma \cdot \frac{L^\gamma}{L^{\gamma+1}} \\ &= \left(1 - \frac{K}{L}\right) \cdot \gamma \end{aligned}$$

Smooth pasting condition then gives

$$\begin{aligned} \left(1 - \frac{K}{L}\right) \cdot \gamma = -1 &\iff 1 + \frac{1}{\gamma} = \frac{K}{L} \\ &\iff L = \frac{\gamma K}{\gamma + 1} \end{aligned}$$

When $a = 0$, then

$$\begin{aligned} \gamma &= \frac{1}{\sigma^2}\mu + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\mu^2 + 2r} \\ &= \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma^2}\sqrt{r^2 - r\sigma^2 + \frac{1}{4}\sigma^4 + 2r\sigma^2} \\ &= \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma^2}\sqrt{r^2 + r\sigma^2 + \frac{1}{4}\sigma^4} \\ &= \frac{1}{\sigma^2}\left[r - \frac{1}{2}\sigma^2 + r + \frac{1}{2}\sigma^2\right] \\ &= \frac{2r}{\sigma^2}, \end{aligned}$$

as expected. Denote

$$L_* = \frac{\gamma K}{\gamma + 1}.$$

We next have that

$$\begin{aligned} \mathrm{d}e^{-rt}v_{L_*}(S(t)) &= e^{-rt}\left[-rv_{L_*}(S(t))\mathrm{d}t + v'_{L_*}(S(t))\mathrm{d}S(t) + \frac{1}{2}v''_{L_*}(S(t))\mathrm{d}S(t)\mathrm{d}S(t)\right] \\ &= e^{-rt}\left[-rv_{L_*}(S(t))\mathrm{d}t + (r - a)S(t)v'_{L_*}(S(t))\mathrm{d}t + \sigma S(t)v'_{L_*}(S(t))\mathrm{d}\tilde{W}(t) + \frac{\sigma^2}{2}v''_{L_*}(S(t))S(t)^2\mathrm{d}t\right] \\ &= e^{-rt}\left[-rv_{L_*}(S(t)) + (r - a)S(t)v'_{L_*}(S(t)) + \frac{\sigma^2}{2}v''_{L_*}(S(t))S(t)^2\right]\mathrm{d}t \\ &+ \sigma S(t)v'_{L_*}(S(t))\mathrm{d}\tilde{W}(t) \end{aligned}$$

When $S(t) < L_*$

$$
\begin{aligned}
-rv_{L_*}(S(t)) + (r-a)S(t)v'_{L_*}(S(t)) + \frac{\sigma^2}{2}v''_{L_*}(S(t))S(t)^2 &= -r(K - S(t)) - (r-a)S(t) \\
&= -rK + aS(t)
\end{aligned}
$$

When $S(t) > L_*$

$$
\begin{aligned}
-rv_{L_*}(S(t)) + (r-a)S(t)v'_{L_*}(S(t)) + \frac{\sigma^2}{2}v''_{L_*}(S(t))S(t)^2 &= L^\gamma \cdot (K-L) \cdot S(t)^{-\gamma}\left[-r + (r-a)\gamma + \frac{\sigma^2}{2}\gamma(\gamma+1)\right] \\
&= 0.
\end{aligned}
$$

Here we used

$$
r + (r-a)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma+1) = 0
$$

To see this, note

$$
\begin{aligned}
r + (r-a)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma+1) = 0 &\iff \frac{1}{2}\sigma^2\gamma^2 - \mu\gamma - r = 0 \\
&\iff \gamma = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\gamma}}{\sigma^2} \text{ or } \frac{\mu - \sqrt{\mu^2 + 2\sigma^2\gamma}}{\sigma^2}.
\end{aligned}
$$

Thus we have shown that

$$
\mathrm{d}e^{-rt}v_{L_*}(S(t)) = \mathbf{1}_{\{S(t)<L_*\}}(-rK + aS(t))\mathrm{d}t + \sigma S(t)v'_{L_*}(S(t))\mathrm{d}\tilde{W}(t).
$$

Continuing

$$
\begin{aligned}
\mathbf{1}_{\{S(t)<L_*\}}(-rK + aS(t)) &\le \mathbf{1}_{\{S(t)<L_*\}}(-rK + aL_*) \\
&= \mathbf{1}_{\{S(t)<L_*\}}\left(-r + \frac{a\gamma}{\gamma+1}\right)K.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-r + \frac{a\gamma}{\gamma+1} \le 0 &\iff a\gamma \le r(\gamma+1) \\
&\iff 0 \le r + \gamma(r-a) \\
&\iff 0 \le \frac{1}{2}\sigma^2\gamma(\gamma+1).
\end{aligned}
$$

But

$$
\begin{aligned}
\gamma &= \frac{1}{\sigma^2}\mu + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\mu^2 + 2r} \\
&\ge \frac{1}{\sigma^2}\mu + \left|\frac{1}{\sigma^2}\mu\right| \\
&\ge 0.
\end{aligned}
$$

Finally, last part follows using the exact same argument as in Corollary 8.3.6. We only need to show that first

$$v_{L_*}(S(t)) \geq (K - S(t))^+ \qquad \text{(Intrinsic Value Bound)}$$

Note

$$0 < L_* \leq K.$$

To see (Intrinsic Value Bound), first consider $S(t) < L_*$. Then $S(t) < K$ and so

$$v_{L_*}(S(t)) = K - S(t) = (K - S(t))^+$$

It remains to show that for $L_* \leq S(t)$ (Intrinsic Value Bound) holds. In other words,

$$(K - L_*) \cdot \frac{L_*^\gamma}{S(t)^\gamma} \geq K - S(t) \iff (K - L_*)L_*^\gamma \geq (K - S(t))\, S(t)^\gamma$$

Consider

$$f(t) = (K - t)t^\gamma.$$

Thus for $t \geq L_*$

$$\begin{aligned} f'(t) &= K\gamma t^{\gamma-1} - (\gamma+1)t^\gamma \\ &= t^{\gamma-1}\left(K\gamma - t(\gamma+1)\right) \\ &\leq t^{\gamma-1}\left(K\gamma - L_*(\gamma+1)\right) \\ &\leq t^{\gamma-1}K\gamma\,(1-1) \\ &= 0. \end{aligned}$$

f is therefore decreasing on $[L_*, +\infty)$ and the desired bound holds. Second, note that v_{L_*} is bounded. Indeed,

$$v_{L_*}(x) \leq K \text{ for all } 0 \leq x \leq L_*$$

Moreover,

$$v_{L_*}(x) \leq K - L_* \text{ for all } x \geq L_*.$$