## Exercise 8.6

Use Optional samplin theorem to show that the following holds

$$\tilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^{+}\right]=\max_{\tau\in\mathcal{T}_{0,T}}\tilde{\mathbb{E}}\left[e^{-r\tau}\left(S(\tau)-K\right)^{+}\right]$$

Each $\tau\in\mathcal{T}_{0,\tau}$ decides to stop at $u\in[0,T]$ based on the path of the stock price between 0 and $u$. This result reiterates the fact that early exercise is useless for American calls paying no dividend.

## Proof

$\tau=T$ is a stopping time which belongs to $\mathcal{T}_{0,T}$. Therefore,

$$\tilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^{+}\right]\leq\max_{\tau\in\mathcal{T}_{0,T}}\tilde{\mathbb{E}}\left[e^{-r\tau}\left(S(\tau)-K\right)^{+}\right]$$

From Lemma 8.5.1., $e^{-rt}\left(S(t)-K\right)^{+}$ is a submartingale under $\tilde{\mathbb{E}}$. Therefore, for each $\tau\in\mathcal{T}_{0,T}$,

$$\tilde{\mathbb{E}}\left[e^{-rT\wedge\tau}\left(S(T\wedge\tau)-K\right)^{+}\right]\leq\tilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^{+}\right]$$

However, for any $\tau\in\mathcal{T}_{0,T}$, $T\wedge\tau=\tau$. Therefore,

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(S(\tau)-K\right)^{+}\right]\leq\tilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^{+}\right],\quad\forall\tau\in\mathcal{T}_{0,T}.$$

Taking max from the left hand side, the result immediately follows.