Exercise 8.7

Suppose that $f(x)$ and $g(x)$ are convex functions. Show that

$$h(x) := \max\{f(x), g(x)\}$$

is also convex.

Proof

Let x_1, x_2 and $\lambda \in [0,1]$. We have that

$$\begin{aligned}\lambda \underbrace{\max\{f(x_1), g(x_1)\}}_{\geq f(x_1)} + (1-\lambda) \underbrace{\max\{f(x_2), g(x_2)\}}_{\geq f(x_2)} &\geq \lambda f(x_1) + (1-\lambda) f(x_2) \\ &\geq f(\lambda x_1 + (1-\lambda) x_2)\end{aligned}$$

Similarly

$$\begin{aligned}\lambda \underbrace{\max\{f(x_1), g(x_1)\}}_{\geq g(x_1)} + (1-\lambda) \underbrace{\max\{f(x_2), g(x_2)\}}_{\geq g(x_2)} &\geq \lambda g(x_1) + (1-\lambda) g(x_2) \\ &\geq g(\lambda x_1 + (1-\lambda) x_2)\end{aligned}$$

Thus,

$$\begin{aligned}\lambda \max\{f(x_1), g(x_1)\} + (1-\lambda) \max\{f(x_2), g(x_2)\} &\geq \max\{f(\lambda x_1 + (1-\lambda)x_2), g(\lambda x_1 + (1-\lambda)x_2)\} \\ &= h(\lambda x_1 + (1-\lambda)x_2)\end{aligned}$$

In other words,

$$\lambda h(x_1) + (1-\lambda) h(x_2) \geq h(\lambda x_1 + (1-\lambda) x_2).$$

h is therefore convex.