Exercise 9.1

Suppose that $M_1(t)$ and $M_2(t)$ are martingales with respect to filtration $\mathcal{F}$ and $M_2 > 0$. Therefore, for $0 \leq s \leq t$, it holds that

$$\mathbb{E}[M_i(t)|\mathcal{F}(s)] = M_i(s) \text{ for } i = 1, 2.$$

Define probability measure $\mathbb{P}^{(M_2)}$ as follows

$$\mathbb{P}^{(M_2)}(A) = \int_A M_2(T) \mathrm{d}\mathbb{P}.$$

Assume $\mathbb{E}M_2 = 1$ to ensure $\mathbb{P}^{(M_2)}(\Omega) = 1$. Show that $\frac{M_1(t)}{M_2(t)}$ is a martingale under $\mathbb{P}^{(M_2)}$.

Proof

Denote expectation under $\mathbb{P}^{(M_2)}$ by $\mathbb{E}^{(M_2)}$. We want to show that

$$\mathbb{E}^{(M_2)}\left[\frac{M_1(t)}{M_2(t)}|\mathcal{F}(s)\right] = \frac{M_1(s)}{M_2(s)}.$$

Let $A \in \mathcal{F}(s)$. We need to show that

$$\int_A \mathbb{E}^{(M_2)}\left[\frac{M_1(t)}{M_2(t)}|\mathcal{F}(s)\right] \mathrm{d}\mathbb{P}^{(M_2)} = \int_A \frac{M_1(s)}{M_2(s)} \mathrm{d}\mathbb{P}^{(M_2)}.$$

On one hand,

$$\begin{aligned}
\int_A \frac{M_1(s)}{M_2(s)} \mathrm{d}\mathbb{P}^{(M_2)} &= \mathbb{E}^{(M_2)}\left[\mathbf{1}_A \frac{M_1(s)}{M_2(s)}\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(s)}{M_2(s)} M_2(T)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_A \frac{M_1(s)}{M_2(s)} M_2(T)|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(s)}{M_2(s)} \mathbb{E}\left[M_2(T)|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(s)}{M_2(s)} M_2(s)\right] \\
&= \mathbb{E}\left[\mathbf{1}_A M_1(s)\right]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_A \mathbb{E}^{(M_2)}\left[\frac{M_1(t)}{M_2(t)}|\mathcal{F}(s)\right] \mathrm{d}\mathbb{P}^{(M_2)} &= \mathbb{E}^{(M_2)}\left[\mathbf{1}_A \mathbb{E}^{(M_2)}\left[\frac{M_1(t)}{M_2(t)}|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}^{(M_2)}\left[\mathbb{E}^{(M_2)}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)}|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}^{(M_2)}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)}\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)} M_2(T)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)} M_2(T)|\mathcal{F}(t)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)} \mathbb{E}\left[M_2(T)|\mathcal{F}(t)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \frac{M_1(t)}{M_2(t)} M_2(t)\right] \\
&= \mathbb{E}\left[\mathbf{1}_A M_1(t)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_A M_1(t)|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A \mathbb{E}\left[M_1(t)|\mathcal{F}(s)\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_A M_1(s)\right]
\end{aligned}$$

Now if S and N are two asset prices and $N > 0$. Then DS and DN are martingales under $\tilde{\mathbb{P}}$. Denote $M_2 = \frac{D(t)N(t)}{N(0)}$ and $M_1(t) = D(t)S(t)$. We have that

$$\begin{aligned}
\mathbb{P}^{(M_2)}(A) &= \int_A M_2(T)\mathrm{d}\tilde{\mathbb{P}} \\
&= \int_A \frac{D(T)N(T)}{N(0)}\mathrm{d}\tilde{\mathbb{P}} \\
&:= \mathbb{P}^{(N)}(A).
\end{aligned}$$

Therefore, $\frac{M_1}{M_2} = \frac{S}{N} := S^{(N)}$ is martingale under $\mathbb{P}^{(M_2)} = \mathbb{P}^{(N)}$.