Exercise 9.2 (Portfolio under change of numéraire

Consider two asset prices $S(t)$ and $N(t)$ along with the money market account as follows:

- $S(t) = S(0)\exp\left(\sigma\tilde{W}(t) + (r - \frac{\sigma^2}{2})t\right)$
- $N(t) = N(0)\exp\left(\nu\tilde{W}(t) + (r - \frac{\nu^2}{2})t\right)$
- $M(t) = e^{rt}$

After denomination of prices in terms of the numéraire N, we obtain $\hat{S}$ and $\hat{M}$. Also, $\hat{W}(t) = \tilde{W}(t) - \nu t$. Consider a portfolio holding $\Delta(t)$ shares of stock at time t and investing in \ borrowing from the money market account to finance it. Denoting the number of money market shares held at time t by $\Gamma(t)$, we have that

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t).$$

- Prove that
$$\mathrm{d}\hat{X}(t) = \Delta(t)\mathrm{d}\hat{S}(t) + \Gamma(t)\mathrm{d}\hat{M}(t).$$
- Compute differentials of $\frac{1}{N(t)}$ and $\hat{M}(t)$.

Proof

- We have that
$$\mathrm{d}\hat{X}(t) = \mathrm{d}X(t)\mathrm{d}N^{-1}(t) + X(t)\mathrm{d}N^{-1}(t) + N^{-1}(t)\mathrm{d}X(t).$$
Computing three terms on the right hand side separately, we obtain that
$$\begin{aligned}
\mathrm{d}X\mathrm{d}N^{-1} &= \Delta\mathrm{d}S\mathrm{d}N^{-1} + \Gamma\mathrm{d}M\mathrm{d}N^{-1} \\
X\mathrm{d}N^{-1} &= \Delta S\mathrm{d}N^{-1} + \Gamma M\mathrm{d}N^{-1} \\
N^{-1}\mathrm{d}X &= \Delta N^{-1}\mathrm{d}S + \Gamma N^{-1}\mathrm{d}M
\end{aligned}$$
Summing up the right-hand sides together, we have that
$$\begin{aligned}
\mathrm{d}\hat{X} &= \Delta[\mathrm{d}S\mathrm{d}N^{-1} + S\mathrm{d}N^{-1} + N^{-1}\mathrm{d}S] + \Gamma[\mathrm{d}M\mathrm{d}N^{-1} + M\mathrm{d}N^{-1} + N^{-1}\mathrm{d}M] \\
&= \Delta \cdot \mathrm{d}\hat{S} + \Gamma\mathrm{d}\hat{M}.
\end{aligned}$$

- We know that
$$\mathrm{d}\hat{S}(t) = (\sigma - \nu)\hat{S}(t)\mathrm{d}\hat{W}(t).$$
When $\sigma = 0$, then $M(t) = S(t)$. Therefore,
$$\mathrm{d}\hat{M}(t) = -\nu\hat{M}(t)\mathrm{d}\hat{W}(t).$$
Next,
$$\begin{aligned}
\mathrm{d}N &= rN\mathrm{d}t + \nu N\mathrm{d}\tilde{W} \\
&= rN\mathrm{d}t + \nu N(\mathrm{d}\hat{W} + \nu\mathrm{d}t) \\
&= (r + \nu^2)N\mathrm{d}t + \nu N\mathrm{d}\hat{W}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}N^{-1} &= -N^{-2}\mathrm{d}N + N^{-3}\mathrm{d}N\mathrm{d}N \\
&= -(r+\nu^2)N^{-1}\mathrm{d}t - \nu N^{-1}\mathrm{d}\hat{W} + \nu^2 N^{-1}\mathrm{d}t \\
&= -N^{-1}\left(r\mathrm{d}t + \nu\mathrm{d}\hat{W}\right)
\end{aligned}$$