Exercise 9.3 (Change in volatility caused by change in numéraire)

Proof

We have that

$$\begin{aligned}
\mathrm{d}\frac{1}{N} &= -N^{-2}\mathrm{d}N + N^{-3}\mathrm{d}N\mathrm{d}N \\
&= -N^{-1}\left[\frac{\mathrm{d}N}{N} - \frac{\mathrm{d}N}{N}\frac{\mathrm{d}N}{N}\right] \\
&= -N^{-1}\left[r\mathrm{d}t + \nu\mathrm{d}\tilde{W}_3 - \nu^2\mathrm{d}t\right] \\
&= -N^{-1}\left[(r-\nu^2)\mathrm{d}t + \nu\mathrm{d}\tilde{W}_3\right]
\end{aligned}$$

Continuing,

$$\begin{aligned}
\mathrm{d}\frac{S}{N} &= \mathrm{d}S\mathrm{d}\frac{1}{N} + S\mathrm{d}\frac{1}{N} + \frac{1}{N}\mathrm{d}S \\
&= \frac{S}{N}\left[-\rho\sigma\nu\mathrm{d}t - (r-\nu^2)S\mathrm{d}t - \nu\mathrm{d}\tilde{W}_3 + r\mathrm{d}t + \sigma\mathrm{d}\tilde{W}_1\right] \\
&= \frac{S}{N}\cdot\left[(\cdots)\mathrm{d}t + \left(\sigma\mathrm{d}\tilde{W}_1 - \nu\mathrm{d}\tilde{W}_3\right)\right]
\end{aligned}$$

Denote by

$$\tilde{W}_4 = \frac{\sigma}{\gamma}\tilde{W}_1 - \frac{\nu}{\gamma}\tilde{W}_3$$

where $\gamma = \sqrt{\sigma^2 - 2\rho\sigma\nu + \nu^2}$. We have that

$$\mathrm{d}\tilde{W}_4\mathrm{d}\tilde{W}_4 = \left(\frac{\sigma^2}{\gamma^2} + \frac{\nu^2}{\gamma^2} - \frac{2\rho\sigma\nu}{\gamma^2}\right)\mathrm{d}t = \mathrm{d}t.$$

Thus, $\tilde{W}_4$ is a Brownian motion. Moreover,

$$\mathrm{d}\frac{S}{N} = \frac{S}{N}\cdot\left[(\cdots)\mathrm{d}t + \gamma\mathrm{d}\tilde{W}_4\right]$$

Define

$$\tilde{W}_2 = \frac{1}{\sqrt{1-\rho^2}}\tilde{W}_3 - \frac{\rho}{\sqrt{1-\rho^2}}\tilde{W}_1$$

Then

$$\mathrm{d}\tilde{W}_2\mathrm{d}\tilde{W}_2 = \left(\frac{1}{1-\rho^2} + \frac{\rho^2}{1-\rho^2} - \frac{2\rho\mathrm{d}\tilde{W}_3\mathrm{d}\tilde{W}_1}{1-\rho^2}\right)\mathrm{d}t = \left(\frac{1-\rho^2}{1-\rho^2}\right)\mathrm{d}t = \mathrm{d}t.$$

Moreover,

$$\mathrm{d}\tilde{W}_1\mathrm{d}\tilde{W}_2 = \frac{1}{\sqrt{1-\rho^2}}\mathrm{d}\tilde{W}_1\mathrm{d}\tilde{W}_3 - \frac{\rho}{\sqrt{1-\rho^2}}\mathrm{d}\tilde{W}_1\mathrm{d}\tilde{W}_1 = \frac{\rho}{\sqrt{1-\rho^2}}\mathrm{d}t - \frac{\rho}{\sqrt{1-\rho^2}}\mathrm{d}t = 0.$$

Therefore, $\tilde{W}_2$ is a Brownian motion independent of $\tilde{W}_1$. See also Exercise 4.13 (Decomposition of correlated Brownian motions into independent Brownian motions).

We next compute the volatility vector of $S^{(N)}(t)$. Note that

$$\begin{aligned}
\mathrm{d}DS &= -rDS\mathrm{d}t + D\mathrm{d}S \\
&= -rDS\mathrm{d}t + rDS\mathrm{d}t + \sigma DS\mathrm{d}\tilde{W}_1 \\
&= \sigma DS\mathrm{d}\tilde{W}_1 \\
&= DS\left[\sigma\mathrm{d}\tilde{W}_1 + 0\mathrm{d}\tilde{W}_2\right]
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{d}DN &= \nu DN\mathrm{d}\tilde{W}_3 \\
&= DN\left[\nu\rho\mathrm{d}\tilde{W}_1 + \nu\sqrt{1-\rho^2}\mathrm{d}\tilde{W}_2\right]
\end{aligned}$$

Volatility vectors of DS and DN are $(\sigma, 0)$ and $(\nu\rho, \nu\sqrt{1-\rho^2})$ resp. Theorem 9.2.2 gives

$$\mathrm{d}S^{(N)} = S^{(N)}\left[\underbrace{(\sigma - \nu\rho)}_{:=\nu_1}\mathrm{d}\tilde{W}_1^{(N)} \underbrace{-\nu\sqrt{1-\rho^2}}_{:=\nu_2}\mathrm{d}\tilde{W}_2^{(N)}\right]$$

Thus,

$$\begin{aligned}
\nu_1^2 + \nu_2^2 &= (\sigma - \nu\rho)^2 + \nu^2(1-\rho^2) \\
&= \sigma^2 + \nu^2\rho^2 - 2\rho\sigma\nu + \nu^2 - \nu^2\rho^2 \\
&= \sigma^2 - 2\rho\sigma\nu + \nu^2.
\end{aligned}$$

Note that volatility of $S^{(N)}$ was determined earlier to be γ. On the other hands, $\sqrt{\nu_1^2 + \nu_2^2}$ is also equal to volatility of $S^{(N)}$. Therefore, $\sqrt{\nu_1^2 + \nu_2^2} = \gamma$.