Exercise 9.4

Use the following equations

$$
\begin{aligned}
\mathrm{d}M(t) &= R(t)M(t)\mathrm{d}t \\
\mathrm{d}S(t) &= S(t)\left[R(t)\mathrm{d}t + \sigma_1(t)\mathrm{d}\tilde{W}_1(t)\right] \\
\mathrm{d}M^f(t)Q(t) &= M^f(t)Q(t)\left[R(t)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\right]
\end{aligned}
$$

To prove

$$
\begin{aligned}
\mathrm{d}\left(\frac{M(t)D^f(t)}{Q(t)}\right) &= -\frac{M(t)D^f(t)}{Q(t)}\sigma_2(t)\mathrm{d}\tilde{W}_3^f(t) \\
\mathrm{d}\left(\frac{D^f(t)S(t)}{Q(t)}\right) &= \frac{S(t)D^f(t)}{Q(t)}\left[\sigma_1(t)\mathrm{d}\tilde{W}_1^f(t) - \sigma_2(t)\mathrm{d}\tilde{W}_3^f(t)\right]
\end{aligned}
$$

Here

$$
\begin{aligned}
\tilde{W}_1^f(t) &= -\int_0^t \sigma_2(u)\rho(u)\mathrm{d}u + \tilde{W}_1(u) \\
\tilde{W}_2^f(t) &= -\int_0^t \sigma_2(u)\sqrt{1-\rho(u)^2}\mathrm{d}u + \tilde{W}_2(u) \\
\tilde{W}_3^f(t) &= -\int_0^t \sigma_2(u)\mathrm{d}u + \tilde{W}_3(u)
\end{aligned}
$$

And $\left(\tilde{W}_1(t), \tilde{W}_2(t)\right)$ and $\left(\tilde{W}_1^f(t), \tilde{W}_2^f(t)\right)$ are independent Brownian motions. Therefore

$$
\begin{aligned}
\mathrm{d}W_1(t)\mathrm{d}W_3(t) &= \rho(t)\mathrm{d}t \\
\mathrm{d}W_2(t)\mathrm{d}W_3(t) &= \sqrt{1-\rho(t)^2}\mathrm{d}t \\
\mathrm{d}\tilde{W}_1^f(t)\mathrm{d}\tilde{W}_3^f(t) &= \rho(t)\mathrm{d}t \\
\mathrm{d}\tilde{W}_2^f(t)\mathrm{d}\tilde{W}_3^f(t) &= \sqrt{1-\rho(t)^2}\mathrm{d}t
\end{aligned}
$$

Proof

Let $X = M^f(t)Q(t)$. Then $\mathrm{d}X = X[\cdots]$. We have that

$$
\begin{aligned}
\mathrm{d}X^{-1} &= -X^{-2}\mathrm{d}X + X^{-3}\mathrm{d}X\mathrm{d}X \\
&= -X^{-1}\left[[\cdots] - \sigma_2^2(t)\mathrm{d}t\right] \\
&= -X^{-1}\left[R(t)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t) - \sigma_2^2(t)\mathrm{d}t\right] \\
&= -X^{-1}\left[\left(R(t) - \sigma_2^2(t)\right)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\mathrm{d}t\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d}\left(\frac{M(t)D^f(t)}{Q(t)}\right) &= \mathrm{d}X^{-1}\mathrm{d}M(t) + M(t)\mathrm{d}X^{-1} + X^{-1}\mathrm{d}M(t) \\
&= M(t)\mathrm{d}X^{-1} + X^{-1}\mathrm{d}M(t) \\
&= -\frac{M(t)}{X}\left[\left(R(t)+\sigma_2^2(t)\right)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\mathrm{d}t\right] + \frac{M(t)}{X}R(t)\mathrm{d}t \\
&= -\frac{M(t)}{X}\left[-\sigma_2^2(t)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\mathrm{d}t\right] \\
&= -\frac{M(t)}{X}\sigma_2(t)\left[-\sigma_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_3(t)\mathrm{d}t\right] \\
&= -\frac{M(t)D^f(t)}{Q(t)}\cdot\sigma_2(t)\mathrm{d}\tilde{W}_3^f(t)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\mathrm{d}\left(\frac{D^f(t)S(t)}{Q(t)}\right) &= \mathrm{d}\left(\frac{S(t)}{X}\right) \\
&= X^{-1}\mathrm{d}S(t) + S(t)\mathrm{d}X^{-1} + \mathrm{d}X^{-1}\mathrm{d}S(t) \\
&= \frac{S(t)}{X}\left[R(t)\mathrm{d}t - \rho(t)\sigma_1(t)\sigma_2(t)\mathrm{d}t + \sigma_1(t)\mathrm{d}\tilde{W}_1(t)\right] \\
&- \frac{S(t)}{X}\left[\left(R(t)-\sigma_2^2(t)\right)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\right] \\
&= \frac{S(t)}{X}\left[\sigma_1(t)\mathrm{d}\tilde{W}_1(t) - \rho(t)\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)\mathrm{d}t - \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\right] \\
&= \frac{S(t)}{X}\left[\sigma_1(t)\left(\mathrm{d}\tilde{W}_1(t) - \rho(t)\sigma_2(t)\right) - \sigma_2(t)\left(-\sigma_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_3(t)\right)\right] \\
&= \frac{S(t)}{X}\left[\sigma_1(t)\mathrm{d}\tilde{W}_1^f(t) - \sigma_2(t)\mathrm{d}\tilde{W}_3^f(t)\right]
\end{aligned}
$$