Exercise 9.5 (Quanto option)

Proof

Under domestic money market measure, it holds that

$$\begin{aligned}
\mathrm{d}S(t) &= S(t)\left[r\mathrm{d}t + \sigma_1 \mathrm{d}\tilde{W}_1(t)\right] \\
\mathrm{d}Q(t) &= Q(t)\left[(r - r^f)\mathrm{d}t + \sigma_2\rho\mathrm{d}\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\mathrm{d}\tilde{W}_2(t)\right] \\
&= Q(t)\left[(r - r^f)\mathrm{d}t + \sigma_2\mathrm{d}\tilde{W}_3(t)\right]
\end{aligned}$$

Here

$$\tilde{W}_3(t) = \int_0^t \rho(u)\mathrm{d}\tilde{W}_1(u) + \int_0^t \sqrt{1-\rho^2(u)}\mathrm{d}\tilde{W}_2(u)$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are independent. We have that

$$\begin{aligned}
\mathrm{d}\log S(t) &= \frac{\mathrm{d}S(t)}{S(t)} - \frac{\mathrm{d}S(t)\mathrm{d}S(t)}{2S^2(t)} \\
&= r\mathrm{d}t + \sigma_1\mathrm{d}\tilde{W}_1(t) - \frac{\sigma_1^2}{2}\mathrm{d}t \\
&= \left(r - \frac{\sigma_1^2}{2}\right)\mathrm{d}t + \sigma_1\mathrm{d}\tilde{W}_1(t)
\end{aligned}$$

Integration gives

$$\log S(t) - \log S(0) = \left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1\tilde{W}_1(t).$$

Thus,

$$S(t) = S(0)e^{\left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1\tilde{W}_1(t)}.$$

Similarly,

$$\begin{aligned}
\mathrm{d}\log Q(t) &= \frac{\mathrm{d}Q(t)}{Q(t)} - \frac{\mathrm{d}Q(t)\mathrm{d}Q(t)}{2Q^2(t)} \\
&= (r - r^f)\mathrm{d}t + \sigma_2\rho\mathrm{d}\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\mathrm{d}\tilde{W}_2(t) - \frac{\sigma_2^2}{2}(\rho^2 + 1 - \rho^2)\mathrm{d}t \\
&= (r - r^f - \frac{\sigma_2^2}{2})\mathrm{d}t + \sigma_2\rho\mathrm{d}\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\mathrm{d}\tilde{W}_2(t)
\end{aligned}$$

Thus,

$$\log Q(t) - \log Q(0) = \left(r - r^f - \frac{\sigma_2^2}{2}\right)t + \sigma_2\rho\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)$$

so that

$$Q(t) = Q(0)e^{\left(r - r^f - \frac{\sigma_2^2}{2}\right)t + \sigma_2\rho\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)}$$

Continuing, we obtain that

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \cdot e^{\left(r-\frac{\sigma_1^2}{2}\right)t+\sigma_1\tilde{W}_1(t)-\left(r-r^f-\frac{\sigma_2^2}{2}\right)t-\sigma_2\rho\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)}$$
$$= \frac{S(0)}{Q(0)} \cdot e^{\left(-\frac{\sigma_1^2}{2}+r^f+\frac{\sigma_2^2}{2}\right)t+(\sigma_1-\sigma_2\rho)\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)}$$

Let

$$\begin{aligned}\sigma_4^2 &= (\sigma_1-\sigma_2\rho)^2+\sigma_2^2(1-\rho^2)\\ &= \sigma_1^2+\sigma_2^2\rho^2-2\sigma_1\sigma_2\rho+\sigma_2^2-\sigma_2^2\rho^2\\ &= \sigma_1^2-2\sigma_1\sigma_2\rho+\sigma_2^2\end{aligned}$$

By Levy theorem

$$\tilde{W}_4(t) = \frac{(\sigma_1-\sigma_2\rho)\,\tilde{W}_1(t)-\sigma_2\sqrt{1-\rho^2}\tilde{W}_2(t)}{\sigma_4}$$

is a Brownian motion. Therefore,

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \cdot e^{\left(-\frac{\sigma_1^2}{2}+r^f+\frac{\sigma_2^2}{2}\right)t+\sigma_4\tilde{W}_4(t)}$$

Finally,

$$\begin{aligned}-\frac{\sigma_1^2}{2}+r^f+\frac{\sigma_2^2}{2} &= -\frac{1}{2}\left(\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho\right)-\sigma_1\sigma_2\rho+r^f+\sigma_2^2\\ &= -\frac{\sigma_4^2}{2}-\sigma_1\sigma_2\rho+r^f+\sigma_2^2\\ &= -\frac{\sigma_4^2}{2}+r-\left(\underbrace{r-r^f+\sigma_1\sigma_2\rho-\sigma_2^2}_{:=a}\right)\end{aligned}$$

Putting pieces together, we have shown that

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \cdot e^{\left(r-a-\frac{\sigma_4^2}{2}\right)t+\sigma_4\tilde{W}_4(t)}$$

Therefore, $\frac{S(t)}{Q(t)}$ satisfies the same equation as of a continuously dividend paying stock with rate a and which has constant volatility σ. Interest rate is constant r. BSM formula from Section 5.5.2 directly applies.