Exercise 9.6

Proof

We begin by noting that

$$\begin{aligned}
\mathrm{d}N(d_\pm) &= N'(d_\pm)\mathrm{d}d_\pm + \tfrac{1}{2}N''(d_\pm)\mathrm{d}d_\pm\mathrm{d}d_\pm \\
&= \frac{1}{\sqrt{2\pi}} \cdot \left[e^{-\frac{d_\pm^2}{2}}\mathrm{d}d_\pm - \frac{d_\pm e^{-\frac{d_\pm^2}{2}}}{2}\mathrm{d}d_\pm\mathrm{d}d_\pm \right] \\
&= \frac{e^{-\frac{d_\pm^2}{2}}}{\sqrt{2\pi}} \cdot \left[\mathrm{d}d_\pm - \frac{d_\pm}{2}\mathrm{d}d_\pm\mathrm{d}d_\pm \right]
\end{aligned}$$

Denote $\tau = T - t$. We obtain that

$$\sigma\sqrt{\tau}d_\pm = \log F_S(t,T) - \log K \pm \frac{\sigma^2}{2}\tau$$

Continuing while keeping in mind that $\mathrm{d}\sqrt{\tau}\mathrm{d}d_\pm = 0$,

$$\begin{aligned}
\sigma\sqrt{\tau}\mathrm{d}d_\pm + \sigma d_\pm \mathrm{d}\sqrt{\tau} &= \frac{1}{F_S(t,T)}\mathrm{d}F_S(t,T) - \frac{1}{2F_S^2(t,T)}\mathrm{d}F_S(t,T)\mathrm{d}F_S(t,T) \pm \frac{\sigma^2}{2}\mathrm{d}\tau \\
&= \sigma \mathrm{d}\tilde{W}^T - \frac{\sigma^2}{2}\mathrm{d}t - \pm\frac{\sigma^2}{2}\mathrm{d}t \\
&= \sigma \mathrm{d}\tilde{W}^T - \sigma^2\left(\frac{1}{2} \pm \frac{1}{2}\right)\mathrm{d}t
\end{aligned}$$

Moreover, since $\mathrm{d}\tau\mathrm{d}\tau = 0$, we have that

$$\mathrm{d}\sqrt{\tau} = \frac{1}{2\sqrt{\tau}}\mathrm{d}\tau = -\frac{1}{2\sqrt{\tau}}\mathrm{d}t.$$

Putting pieces together, we obtain that

$$\mathrm{d}d_\pm = \frac{1}{\sqrt{\tau}}\left[\mathrm{d}\tilde{W}^T - \left[\sigma\left(\frac{1}{2} \pm \frac{1}{2}\right) - \frac{d_\pm}{2\sqrt{\tau}}\right]\mathrm{d}t\right]$$

In particular,

$$\mathrm{d}d_-\mathrm{d}d_- = \mathrm{d}d_+\mathrm{d}d_+ = \frac{1}{\tau}\mathrm{d}t \text{ and } \mathrm{d}F\mathrm{d}N(d_\pm) = \frac{e^{-\frac{d_\pm^2}{2}}}{\sqrt{2\pi}} \cdot \frac{\sigma F}{\sqrt{\tau}}\mathrm{d}t$$

Moreover, since $d_+ - d_- = \sigma\sqrt{\tau}$

$$\mathrm{d}d_+ - \mathrm{d}d_- = \frac{1}{\sqrt{\tau}}\left[-\sigma + \frac{d_+}{2\sqrt{\tau}}\right]\mathrm{d}t - \frac{1}{\sqrt{\tau}}\left[\frac{d_-}{2\sqrt{\tau}}\right]\mathrm{d}t = \frac{-\sigma}{2\sqrt{\tau}}$$

We will then have that

$$F\mathrm{d}N(d_+) + \mathrm{d}F\mathrm{d}N(d_+) - K\mathrm{d}N(d_-) = \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} \cdot \left[F\mathrm{d}d_+ - \frac{Fd_+}{2\tau}\mathrm{d}t + \frac{\sigma F}{\sqrt{\tau}}\mathrm{d}t - Ke^{\frac{d_+^2 - d_-^2}{2}}\left[\mathrm{d}d_- - \frac{d_-}{2\tau}\mathrm{d}t\right]\right]$$

On the other hand,

$$d_+ - d_- = \sigma\sqrt{\tau} \text{ and } d_+ + d_- = \frac{2\left[\log F_S(t,T) - \log K\right]}{\sigma\sqrt{\tau}}$$

Therefore,

$$d_+^2 - d_-^2 = (d_+ + d_-)(d_+ - d_-) = 2\left(\log F_S(t,T) - \log K\right)$$

Thus,

$$e^{\frac{d_+^2 - d_-^2}{2}} = \frac{F}{K}.$$

Putting pieces together

$$F\mathrm{d}N(d_+) + \mathrm{d}F\mathrm{d}N(d_+) - K\mathrm{d}N(d_-) = \frac{Fe^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} \cdot \left[\mathrm{d}d_+ - \frac{d_+}{2\tau}\mathrm{d}t + \frac{\sigma}{\sqrt{\tau}}\mathrm{d}t - \mathrm{d}d_- + \frac{d_-}{2\tau}\mathrm{d}t\right]$$

Continuing

$$\begin{aligned}
\mathrm{d}d_+ - \frac{d_+}{2\tau}\mathrm{d}t + \frac{\sigma}{\sqrt{\tau}}\mathrm{d}t - \mathrm{d}d_- + \frac{d_-}{2\tau}\mathrm{d}t &= (\mathrm{d}d_+ - \mathrm{d}d_-) - \frac{1}{2\tau}\left(d_+ - d_-\right)\mathrm{d}t + \frac{\sigma}{\sqrt{\tau}}\mathrm{d}t \\
&= -\frac{\sigma}{2\sqrt{\tau}} - \frac{\sigma}{2\sqrt{\tau}} + \frac{\sigma}{2\tau}\mathrm{d}t \\
&= 0.
\end{aligned}$$